Geometry and Topology: Individual

(1) Let $M = \{(x^1, y^1, \cdots, x^n, y^n) \mid \sum_{i=1}^n (x^i)^2 = 1, \sum_{i=1}^n x^i y^i = 0\} \subset \mathbb{R}^{2n}$. Show that

(a) M is a smooth manifold and a vector bundle over the $(n-1)$ dimensional sphere. Compute the Euler class of this vector bundle.

(b) M is a symplectic manifold, i.e. there exists a non-degenerate closed 2-form on M.

(2) Let f be a smooth function on $\mathbb{R}^n$ that satisfies $|\nabla f| < 1$ and f vanishes at the origin, and let M be the graph of f in $\mathbb{R}^{n+1}$ with standard coordinates $x^1, \cdots, x^{n+1}$. Show that the function

$$g = -(x^{n+1})^2 + \sum_{i=1}^n (x^i)^2$$

restricts to a proper function on M, i.e for any $c > 0$, the intersection of $g^{-1}((-\infty, c])$ with M is always compact.

(3) Let X and Y be two compact Riemann surfaces with Euler characteristics $\chi(X)$ and $\chi(Y)$, respectively. Suppose $\chi(X) > \chi(Y)$, prove that there exists no non-trivial holomorphic map from X to Y.

(4) Show that a complete surface in $\mathbb{R}^3$ with finite area and negative curvature has at least four ends.